

Norm-constrained determinantal representations of multivariable polynomials

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Abstract. For every multivariable polynomial p , with $p(0) = 1$, we construct a determinantal representation

$$p = \det(I - KZ),$$

where Z is a diagonal matrix with coordinate variables on the diagonal and K is a complex square matrix. Such a representation is equivalent to the existence of K whose principal minors satisfy certain linear relations. When norm constraints on K are imposed, we give connections to the multivariable von Neumann inequality, Agler denominators, and stability. We show that if a multivariable polynomial q , $q(0) = 0$, satisfies the von Neumann inequality, then $1 - q$ admits a determinantal representation with K a contraction. On the other hand, every determinantal representation with a contractive K gives rise to a rational inner function in the Schur–Agler class.

1. Introduction

Our object of study is determinantal representations

$$p(z) = \det(I_{|n|} - KZ_n), \tag{1.1}$$

for a d -variable polynomial $p(z)$, $z = (z_1, \dots, z_d)$, with $p(0) = 1$. Here $n = (n_1, \dots, n_d)$ is in the set \mathbb{N}_0^d of d -tuples of nonnegative integers, $|n| = n_1 + \dots + n_d$, $Z_n = \bigoplus_{i=1}^d z_i I_{n_i}$, and K is a complex square matrix. It is of interest of how and to what extent, the algebraic and operator-theoretic properties of the polynomial correspond to the size and norm of the matrix K of its representation.

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Various determinantal representations of polynomials have been studied, often for polynomials over the reals: see a recent overview article [27], together with bibliography, and also [21]. The particular form of (1.1) has appeared before, for instance, in [6]. An important early result on two-variable polynomials was obtained by A. Kummert [19, Theorem 1].

Given a d -variable polynomial $p(z)$, $p(0) = 1$, we consider the question of whether it can be represented in the form (1.1) for some $n \in \mathbb{N}_0^d$ and some $|n| \times |n|$ complex matrix K , possibly subject to a constraint. It will be shown that the unconstrained version of this question can always be answered in the affirmative (Section 2). The problem of minimizing the operator norm of K over all representations (1.1) of p will be seen to be more involved (Section 3).

We will say that the *multi-degree* $\deg p$ of a polynomial p is $m = (m_1, \dots, m_d)$ if $m_i = \deg_i p$ is the degree of p as a polynomial of z_i , $i = 1, \dots, d$. The *total degree* $\text{tdeg } p$ of p is the largest $|k|$ over all monomials $z^k = z_1^{k_1} \dots z_d^{k_d}$ of p . For $m, n \in \mathbb{N}_0^d$, the inequality $m \leq n$ will be meant in the usual component-wise sense: $m_i \leq n_i$, $i = 1, \dots, d$.

For a matrix K , the principal submatrix determined by an index set α will be denoted by $K[\alpha]$. Given a collection of complex numbers c_α , indexed by nonempty subsets α of $\{1, \dots, d\}$, the Principal Minor Assignment Problem (see, e.g., [25, 15]) consists of finding a $d \times d$ matrix K such that $\det K[\alpha] = c_\alpha$ for all α . This problem is, in general, overdetermined since the number of independent principal minors grows exponentially with the matrix size, d , while the number of free parameters, the matrix entries, is d^2 . It becomes well-posed under additional assumptions on K or d . For theoretical and computational advances, see [12, 20, 16].

A polynomial of multi-degree $(1, \dots, 1)$, is said to be *multi-affine*. For such a polynomial, the problem of finding a representation (1.1) with $n = (1, \dots, 1)$ is equivalent to the Principal Minor Assignment Problem. This follows by comparing the expansion

$$\det(I_d - KZ_{(1, \dots, 1)}) = 1 + \sum_{\alpha \neq \emptyset} (-1)^{\text{card } \alpha} \det K[\alpha] \prod_{i \in \alpha} z_i,$$

to the general form

$$p(z) = 1 + \sum_{\alpha \neq \emptyset} (-1)^{\text{card } \alpha} c_\alpha \prod_{i \in \alpha} z_i$$

of a d -variable multi-affine polynomial p , $p(0) = 1$.

For a general polynomial p , finding a determinantal representation (1.1) with $n = \deg p$ may not be possible by the same dimension count as above. It is clear that (1.1) implies that $n \geq \deg p$. If n is prescribed, one may view (1.1) as the Principal Minor Relation Problem formulated in Section 2.

This paper is largely motivated by our study [13] of the multivariable von Neumann inequality and the discrepancy between the Schur and Schur-Agler norms of analytic functions on the unit polydisk

$$\mathbb{D}^d = \{z = (z_1, \dots, z_d) \in \mathbb{C}^d : |z_i| < 1, i = 1, \dots, d\}.$$

The *Schur class*, consisting of analytic $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ -valued functions f on \mathbb{D}^d such that

$$\|f\|_\infty := \sup_{z \in \mathbb{D}^d} \|f(z)\| \leq 1, \quad (1.2)$$

will be denoted by $\mathcal{S}_d(\mathcal{U}, \mathcal{Y})$. Here $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ is the Banach space of bounded linear operators from a Hilbert space \mathcal{U} to a Hilbert space \mathcal{Y} . The *Schur–Agler class*, introduced in [1], will be denoted by $\mathcal{SA}_d(\mathcal{U}, \mathcal{Y})$. It consists of analytic $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ -valued functions on \mathbb{D}^d such that

$$\|f\|_{\mathcal{A}} := \sup_T \|f(T)\| \leq 1, \quad (1.3)$$

where the supremum is taken over all d -tuples $T = (T_1, \dots, T_d)$ of commuting strict contractions on a common Hilbert space. In the scalar case $\mathcal{U} = \mathcal{Y} = \mathbb{C}$ and in the case $\mathcal{U} = \mathcal{Y}$, we will use respective shortcuts \mathcal{S}_d , \mathcal{SA}_d , and $\mathcal{S}_d(\mathcal{U})$, $\mathcal{SA}_d(\mathcal{U})$.

For a bounded analytic function $f: \mathbb{D}^d \rightarrow \mathcal{L}(\mathcal{U}, \mathcal{Y})$, the *von Neumann inequality* is the inequality between its Schur and Schur–Agler norms:

$$\|f\|_{\mathcal{A}} \leq \|f\|_\infty. \quad (1.4)$$

It is valid when $d = 1$ [28] and $d = 2$ [2], and not always valid when $d \geq 3$ [26, 8, 14]. Thus a Schur function f is Schur–Agler if and only if (1.4) holds. One has the inclusion $\mathcal{SA}_d(\mathcal{U}, \mathcal{Y}) \subseteq \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$. The two classes coincide when $d = 1$ and $d = 2$, and the inclusion is proper when $d \geq 3$. See, e.g., [13] for details.

A d -variable polynomial is said to be *stable* if it has no zeros in $\overline{\mathbb{D}}^d$, and *semi-stable* if it has no zeros in \mathbb{D}^d . A rational function in $\mathcal{S}_d(\mathbb{C}^N)$ is said to be *inner* if its radial limits are unitary (unimodular, in the scalar case) almost everywhere on the d -torus. Every scalar-valued rational inner function is necessarily of the form $f(z) = z^n \bar{p}(1/z)/p(z)$ for some $n \in \mathbb{N}_0^d$ and a semi-stable polynomial p [24, Theorem 5.5.1], where $\bar{p}(z) := \overline{p(\bar{z})}$. A rational inner function $f \in \mathcal{S}_d$ is said to have a *transfer-function realization* (of order $m \in \mathbb{N}_0^d$) if there exists a unitary matrix

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{C}^{(1+|m|) \times (1+|m|)}$$

so that

$$f(z) = A + BZ_m(I - DZ_m)^{-1}C. \quad (1.5)$$

Such a realization for a scalar-valued rational inner f exists if and only if $f \in \mathcal{SA}_d$ [1], [17, Theorem 2.9].

In Section 4, we explore the Schur–Agler class in the context of exterior products, proving, in particular, that if S is a matrix-valued Schur–Agler function, then so are its determinant $\det S$ and permanent $\text{per } S$.

Following [18], we will say that a semi-stable polynomial p is an *Agler denominator* if the rational inner function $z^{\deg p} \bar{p}(1/z)/p(z)$ is Schur–Agler. Extending this notion, we will call a semi-stable polynomial p an *eventual Agler denominator* of order $n \in \mathbb{N}_0^d$ if $z^n \bar{p}(1/z)/p(z)$ is Schur–Agler.

Representations (1.1) may allow for a fresh approach to the study of the multivariable von Neumann inequality (1.4). In Section 5, we examine the discrepancy between the Schur and Schur–Agler classes via (eventual) Agler denominators. It is shown that (i) not every (semi-)stable polynomial is an Agler denominator, (ii) if q is a polynomial in \mathcal{SA}_d with $q(0) = 0$, then $1 - q$ admits a representation (1.1) for some $n \in \mathbb{N}_0^d$ and K a contraction, and (iii) (building on results in Section 4) if p is representable in the form (1.1) for some $n \in \mathbb{N}_0^d$ and contractive K , then p is an eventual Agler denominator of order n . As a corollary, we deduce that every semi-stable linear polynomial is an Agler denominator, thus solving a problem suggested in [18]. To illustrate a possible advantage of our approach, we compare a minimal determinantal representation (1.1) to a minimal transfer-function realization (1.5) in Remark 5.10.

In Section 6, we revisit the Kaijser–Varopoulos–Holbrook example to build a family of polynomials in $\mathcal{S}_d \setminus \mathcal{SA}_d$, for every odd $d \geq 3$. If $d = 3$, this leads to a slightly improved bound for the von Neumann constant.

2. Unconstrained determinantal representations

Theorem 2.1. *Every $p \in \mathbb{C}[z_1, \dots, z_d]$, with $p(0) = 1$, admits a representation (1.1) for some $n \in \mathbb{N}_0^d$ and some $K \in \mathbb{C}^{|n| \times |n|}$.*

Note that the d -tuple n is not prescribed in the statement, although a bound on n will be deduced in the proof. If n is specified, as in the Principal Minor Assignment Problem, Theorem 2.1 ensures the solvability of the following problem for some n (see also Remark 3.8).

The Principal Minor Relation Problem. Let $m \in \mathbb{N}_0^d$, and let p_k , $0 \leq k \leq m$, be a collection of complex numbers. Given $n = (n_1, \dots, n_d) \in \mathbb{N}_0^d$, $n \geq m$, find a matrix $K \in \mathbb{C}^{|n| \times |n|}$ whose principal minors $K[\alpha_1 \cup \dots \cup \alpha_d]$, indexed by $\alpha_1 \subseteq \{1, \dots, n_1\}$, $\alpha_2 \subseteq \{n_1 + 1, \dots, n_1 + n_2\}$, \dots , $\alpha_d \subseteq \{n_1 + \dots + n_{d-1} + 1, \dots, |n|\}$, satisfy the relations

$$(-1)^{|k|} \sum_{|\alpha_i|=k_i, i=1, \dots, d} \det K[\alpha_1 \cup \dots \cup \alpha_d] = p_k, \quad 0 \leq k \leq m. \quad (2.1)$$

When $m = n = (1, \dots, 1)$, this is the classical Principal Minor Assignment Problem mentioned in Section 1.

The following standard result (see, e.g., [23, Theorem 3.1.1]) will occasionally be used.

Lemma 2.2. *Let $P = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a block matrix with square matrices A and D . If $\det A \neq 0$, then $\det P = \det A \det(D - CA^{-1}B)$. Similarly, if $\det D \neq 0$, then $\det P = \det D \det(A - BD^{-1}C)$.*

The proof of Theorem 2.1 will be based on the next two lemmas.

Lemma 2.3. *For every $q \in \mathbb{C}^{a \times b}[z_1, \dots, z_d]$, there exist natural numbers $s_0 = a, s_1, \dots, s_{t-1}, s_t = b$, matrices $C_i \in \mathbb{C}^{s_i \times s_{i+1}}$, and diagonal $s_i \times s_i$ matrix functions L_i with the diagonal entries in $\{1, z_1, \dots, z_d\}$, such that*

$$q(z) = C_0 L_1(z) \cdots C_{t-1} L_t(z) C_t. \quad (2.2)$$

The factorization can be chosen so that $t = \text{tdeg } q$.

Proof. We apply induction on t . If $t = 0$, then (2.2) holds trivially with $C_0 = q(z)$. Suppose a representation (2.2) exists for every matrix polynomial in z_1, \dots, z_d of total degree $t - 1$. Then a polynomial $q \in \mathbb{C}^{a \times b}[z_1, \dots, z_d]$ of total degree t can be represented in the form

$$\begin{aligned} q(z) &= q_0 + z_1 q_1(z) + \cdots + z_d q_d(z) \\ &= \begin{bmatrix} q_0 & q_1(z) & \cdots & q_d(z) \end{bmatrix} \begin{bmatrix} I_b & 0 & \cdots & 0 \\ 0 & z_1 I_b & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & z_d I_b \end{bmatrix} \begin{bmatrix} I_b \\ I_b \\ \vdots \\ I_b \end{bmatrix}, \end{aligned}$$

where $\begin{bmatrix} q_0 & q_1(z) & \cdots & q_d(z) \end{bmatrix} \in \mathbb{C}^{a \times (d+1)b}[z_1, \dots, z_d]$ is a polynomial of total degree $t - 1$. By assumption, we have

$$\begin{bmatrix} q_0 & q_1(z) & \cdots & q_d(z) \end{bmatrix} = C_0 L_1(z) \cdots C_{t-2} L_{t-1}(z) C_{t-1},$$

which gives $q(z) = C_0 L_1(z) \cdots C_{t-1} L_t(z) C_t$, with

$$L_t = \begin{bmatrix} I_b & 0 & \cdots & 0 \\ 0 & z_1 I_b & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & z_d I_b \end{bmatrix}, \quad C_t = \begin{bmatrix} I_b \\ I_b \\ \vdots \\ I_b \end{bmatrix}.$$

□

Lemma 2.4. *Let $A_i \in \mathbb{C}^{s_i \times s_{i+1}}$, $i = 0, \dots, t - 1$, and $A_t \in \mathbb{C}^{s_t \times s_0}$, where $s_0 = a$. Then*

$$\det \begin{bmatrix} I_a & -A_0 & 0 & \cdots & 0 \\ 0 & I_{s_1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & -A_{t-1} \\ -A_t & 0 & \cdots & 0 & I_{s_t} \end{bmatrix} = \det(I_a - A_0 \cdots A_t). \quad (2.3)$$

Proof. We apply induction on $t \geq 1$. For $t = 1$, Lemma 2.2 gives

$$\det \begin{bmatrix} I_a & -A_0 \\ -A_1 & I_{s_1} \end{bmatrix} = \det(I_a - A_0 A_1).$$

Suppose (2.3) holds for $t - 1$ in the place of t . Then, again by Lemma 2.2,

$$\begin{aligned}
 & \det \begin{bmatrix} I_a & -A_0 & 0 & \dots & 0 \\ 0 & I_{s_1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -A_{t-2} & 0 \\ 0 & \dots & 0 & I_{s_{t-1}} & -A_{t-1} \\ -A_t & 0 & \dots & 0 & I_{s_t} \end{bmatrix} \\
 &= \det \left(\begin{bmatrix} I_a & -A_0 & 0 & \dots & 0 \\ 0 & I_{s_1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & -A_{t-2} \\ 0 & \dots & \dots & 0 & I_{s_{t-1}} \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -A_{t-1} \end{bmatrix} [-A_t \ 0 \ \dots \ 0] \right) \\
 &= \det \begin{bmatrix} I_a & -A_0 & 0 & \dots & 0 \\ 0 & I_{s_1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & -A_{t-2} \\ -A_{t-1}A_t & 0 & \dots & 0 & I_{s_{t-1}} \end{bmatrix} \\
 &= \det(I_a - A_0 \cdots A_{t-1}A_t).
 \end{aligned}$$

□

Proof of Theorem 2.1. Applying Lemma 2.3 to $q = 1 - p$, we obtain

$$p(z) = 1 - C_0 L_1(z) \cdots C_{t-1} L_t(z) C_t$$

(here $a = b = 1$). So, by Lemma 2.4,

$$p(z) = \det(I_N - Q(z)),$$

where $N = 1 + s_1 + \cdots + s_t$ and

$$Q(z) = \begin{bmatrix} 0 & C_0 L_1(z) & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & C_{t-1} L_t(z) \\ C_t & 0 & \dots & \dots & 0 \end{bmatrix}.$$

Since $Q(z)$ factors as $C \cdot L(z)$, where

$$C = \begin{bmatrix} 0 & C_0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & C_{t-1} \\ C_t & 0 & \dots & \dots & 0 \end{bmatrix}, \quad L(z) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & L_1(z) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & L_t(z) \end{bmatrix}, \quad (2.4)$$

it may be written in the form

$$Q(z) = TGT^{-1} \cdot T \begin{bmatrix} I_{N-|n|} & 0 \\ 0 & Z_n \end{bmatrix} T^{-1}$$

where $G = T^{-1}CT \in \mathbb{C}^{N \times N}$ and T is a permutation matrix. Representing G as a 2×2 block matrix, we obtain that

$$p(z) = \det \begin{bmatrix} I_{N-|n|} - G_{11} & -G_{12}Z_n \\ -G_{21} & I_{|n|} - G_{22}Z_n \end{bmatrix}.$$

Hence $\det(I_{N-|n|} - G_{11}) = p(0) = 1$ and, in particular, the matrix $I_{N-|n|} - G_{11}$ is invertible. Therefore, by Lemma 2.2,

$$p(z) = \det(I_{|n|} - G_{22}Z_n - G_{21}(I_{N-|n|} - G_{11})^{-1}G_{12}Z_n) = \det(I_{|n|} - KZ_n)$$

with $K = G_{22} + G_{21}(I_{N-|n|} - G_{11})^{-1}G_{12}$. \square

3. Constrained determinantal representations

We will now look into the existence of a determinantal representation (1.1) with a norm constraint on the matrix K . First, we give norm-constrained versions of Lemma 2.3 and Theorem 2.1.

Lemma 3.1. *For every polynomial $q \in \mathcal{SA}_d(\mathbb{C}^b, \mathbb{C}^a)$, a factorization (2.2) exists with constant contractive matrices C_i , $i = 0, \dots, t$, where $t \geq \text{tdeg } q$.*

Proof. Let $q \in \mathcal{SA}_d(\mathbb{C}^b, \mathbb{C}^a)$ be a polynomial. By [22, Corollary 18.2], q can be written as a product of constant contractive matrices and diagonal matrices with monomials on the diagonal. Every such diagonal matrix is, in turn, a product of matrices L_i as in (2.2) (interlacing with $C_i = I$). \square

Theorem 3.2. *Let p be a polynomial of the form $p(z) = \det(I_N - q(z))$, where q is a Schur–Agler polynomial with matrix coefficients, i.e., $q \in \mathbb{C}^{N \times N}[z_1, \dots, z_d]$ and $q \in \mathcal{SA}_d(\mathbb{C}^N)$. If $p(0) = 1$, then (1.1) holds with K a contraction.*

Proof. By Lemma 3.1, the matrices C_i in the factorization (2.2) can be chosen contractive. Then the matrix G as in the proof of Theorem 2.1 is also contractive, and by the standard closed-loop mapping argument, K is contractive as well. For reader's convenience, we include this argument.

Given $u \in \mathbb{C}^{|n|}$, the vector equation

$$\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

in $x \in \mathbb{C}^{N-|n|}$ and $y \in \mathbb{C}^{|n|}$ has a unique solution $x = (I_{N-|n|} - G_{11})^{-1}G_{12}u$, $y = (G_{22} + G_{21}(I_{N-|n|} - G_{11})^{-1}G_{12})u = Ku$. Since G is a contraction, we have $\|x\|^2 + \|y\|^2 \leq \|x\|^2 + \|u\|^2$, i.e., $\|y\| \leq \|u\|$. Since $u \in \mathbb{C}^{|n|}$ is arbitrary, K is a contraction as claimed. \square

Corollary 3.3. *Let $p \in \mathbb{C}[z_1, \dots, z_d]$, with $p(0) = 1$, be such that $q = 1 - p \in \mathcal{SA}_d$, then (1.1) holds with K a contraction.*

Remark 3.4. The converse to Corollary 3.3 is false. Indeed, if $d = 1$ and $\sqrt{2} - 1 < |a| \leq 1$, then $p(z) = (1 - az)^2$ satisfies (1.1) with $K = aI_2$, obviously a contraction. However, $\|1 - p\|_{\mathcal{A}} = \|1 - p\|_{\infty} = 2|a| + |a|^2 > 1$.

Define the *stability radius* $s(p)$ of a d -variable polynomial p to be

$$s(p) = \sup \left\{ r > 0 : p(z) \neq 0, z \in r\mathbb{D}^d \right\}.$$

Clearly, p is semi-stable if $s(p) \geq 1$, and stable if $s(p) > 1$. It is easy to see that $\|K\| \geq 1/s(p)$ whenever p admits (1.1).

Remark 3.5. With respect to a given subalgebra $\Delta \subseteq \mathbb{C}^{d \times d}$, the structured singular value $\mu_{\Delta}(K)$ of a matrix $K \in \mathbb{C}^{d \times d}$ is defined to be

$$\mu_{\Delta}(K) := (\inf \{ \|Z\| : Z \in \Delta \text{ and } \det(I - KZ) = 0 \})^{-1}.$$

The theory of structured singular values was introduced in [9] to analyze linear systems with structured uncertainties; for an overview, see for instance [29, Chapter 10]. If p satisfies (1.1) and $\Delta = \{Z_n = \bigoplus_{i=1}^d z_i I_{n_i} : z \in \mathbb{C}^d\}$, we recognize that $\mu_{\Delta}(K) = 1/s(p)$.

The next theorem gives a way of constructing a representation (1.1) with a certain upper bound on the norm of K .

Theorem 3.6. *Given a polynomial $p(z) = 1 + \sum_{k \in S} p_k z^k$, where $S \subseteq \mathbb{N}_0^d \setminus \{0\}$ and the coefficients $p_k, k \in S$, are nonzero, let $t = \text{tdeg } p$, $n = \sum_{k \in S} k$, and $\beta = (\sum_{k \in S} |p_k|)^{\frac{1}{t+1}}$. Then p admits a representation (1.1) with $K \in \mathbb{C}^{|n| \times |n|}$, and*

$$\|K\| \leq \beta \max \left\{ \sqrt{(\beta^2 - 1)(1 + \beta + \dots + \beta^{\kappa-1})^2 + 1}, 1 \right\} \quad (3.1)$$

for some integer κ , $1 \leq \kappa \leq t$.

Remark 3.7. If $\beta \leq 1$ and $s(p) = 1/\beta$, which is the case for semi-stable linear polynomials, the norm bound asserted in the theorem is sharp. In general, it is not sharp, even in the univariate case.

Remark 3.8. Theorem 3.6 implies that the Principal Minor Relation Problem (2.1) with data $\{p_k \neq 0 : k \in S\}$ is solvable for $n = \sum_{k \in S} k$.

Proof of Theorem 3.6. Form the matrices

$$C_0 = -\beta^{\frac{1-t}{2}} \text{row}_{k \in S} \left[|p_k|^{\frac{1}{2}} \right], \quad C_1 = \dots = C_{t-1} = \beta I_{|S|}, \quad C_t = \beta^{\frac{1-t}{2}} \text{col}_{k \in S} \left[\frac{p_k}{|p_k|^{\frac{1}{2}}} \right],$$

all of equal norm β . Relative to the standard ordering of factors,

$$z^k = \underbrace{z_1 \cdots z_1}_{k_1 \text{ times}} \cdot \underbrace{z_2 \cdots z_2}_{k_2 \text{ times}} \cdots \underbrace{z_d \cdots z_d}_{k_d \text{ times}},$$

write each monomial z^k , $k \in S$, as an expanded product $z^k = z_{i_1(k)} \cdots z_{i_t(k)}$, where $z_{i_j(k)} \neq 1$, for $1 \leq j \leq |k|$, and $z_{i_j(k)} = 1$, for $|k| + 1 \leq j \leq t$. Let

$$L_j(z) = \text{diag}[z_{i_j(k)}], \quad j = 1, \dots, t,$$

and observe that $L_1(z)$ contains no unit entries by construction. It is then easy to check that (2.2) holds for $q = 1 - p$. Thus, by Lemma 2.4, we obtain

$$\begin{aligned} p(z) &= \det \left(I_{1+|S|t} - \begin{bmatrix} 0 & C_0 L_1(z) & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & C_{t-1} L_t(z) \\ C_t & 0 & \dots & \dots & 0 \end{bmatrix} \right) \\ &= \det(I_{1+|S|t} - C \cdot L(z)), \end{aligned}$$

where C and $L(z)$ are as in (2.4).

Using an appropriate permutation T , we can bubble-sort $L(z)$ so that all diagonal ones are stacked in the left upper corner block:

$$TL(z)T^{-1} = \begin{bmatrix} I_\ell & 0 \\ 0 & Z_n \end{bmatrix},$$

where $\ell = 1 + |S|t - |n|$. Then, partitioned accordingly,

$$TCT^{-1} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} =: G$$

has the following structure:

$$G = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & * & 0 & 0 & \dots & 0 \\ \vdots & \ddots & * & \ddots & \vdots & 0 & 0 & * & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & * & \vdots & & \ddots & \ddots & * \\ * & 0 & \dots & \dots & 0 & 0 & \dots & \dots & 0 & 0 \\ \hline 0 & * & 0 & \dots & 0 & 0 & * & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & * & \vdots & & \ddots & \ddots & * \\ * & 0 & \dots & \dots & 0 & 0 & \dots & \dots & \dots & 0 \end{bmatrix}.$$

We observe that G_{11} is nilpotent of index κ , where κ is the number of nonzero blocks $L_i(0)$ (counting $L_0 \equiv 1$). Since, $L_1(0) = 0$, we necessarily have $1 \leq \kappa \leq t$. Thus, we obtain that

$$p(z) = \det(I_{1+|S|t} - C \cdot L(z)) = \det \left(I_{1+|S|t} - G \cdot \begin{bmatrix} I_\ell & 0 \\ 0 & Z_n \end{bmatrix} \right) = \det(I_{|n|} - K Z_n),$$

where $K = G_{22} + G_{21}(I - G_{11})^{-1}G_{12}$ by the same argument as in the proof of Theorem 3.2.

The norm bound on K is obtained as follows. For a fixed $u \in \mathbb{C}^{|n|}$, the solution to the vector equation

$$\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

in $x \in \mathbb{C}^\ell$ and $y \in \mathbb{C}^{|n|}$ is given by

$$x = (I_\ell - G_{11})^{-1} G_{12} u = (I_\ell + G_{11} + \cdots + G_{11}^{\kappa-1}) G_{12} u, \quad y = Ku.$$

Observing that $\|G\| = \beta$, we have

$$\|x\|^2 + \|y\|^2 \leq \beta^2 (\|x\|^2 + \|u\|^2).$$

If $\beta \leq 1$, then $\|y\| \leq \beta \|u\|$ and thus $\|K\| \leq \beta$. If $\beta > 1$, then the estimate

$$\|x\| \leq (1 + \beta + \cdots + \beta^{\kappa-1}) \beta \|u\|$$

implies that

$$\|y\|^2 \leq (\beta^2 - 1) \|x\|^2 + \beta^2 \|u\|^2 \leq \beta^2 ((\beta^2 - 1)(1 + \beta + \cdots + \beta^{\kappa-1})^2 + 1) \|u\|^2,$$

which yields (3.1). \square

Remark 3.9. Given a d -variable polynomial p , $p(0) = 1$, and a d -tuple $n \geq \deg p$ such that (1.1) holds, one may consider the set $\mathcal{K}_n(p)$ of $|n| \times |n|$ matrices K such that $\det(I_{|n|} - KZ_n) = p(z)$. It is then of interest to determine the constant

$$\alpha(p) := \inf_n \min_{K \in \mathcal{K}_n(p)} \|K\|.$$

In particular, it is unclear whether $\alpha(p) < 1$ ($\alpha(p) \leq 1$) for p stable (semi-stable).

4. The Schur–Agler class and wedge powers

We will now examine the Schur–Agler norm of tensor and exterior products of operator-valued functions. The results are preceded by some definitions. For a background on tensor and exterior algebras see, e.g., [4, 10, 11].

Let $\mathcal{V}^{\otimes k}$ be the k -fold tensor power of a vector space \mathcal{V} . The k -th antisymmetric tensor power $\mathcal{V}^{\wedge k}$ of \mathcal{V} may be viewed as a subspace of $\mathcal{V}^{\otimes k}$, generated by elementary antisymmetric tensors

$$v_1 \wedge \cdots \wedge v_k = \sum_{\sigma} (\text{sign } \sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)},$$

where the summation is taken over all permutations σ of $1, 2, \dots, k$.

Given a linear map $A : \mathcal{U} \rightarrow \mathcal{V}$ of vector spaces, the linear operator $A^{\wedge k} : \mathcal{U}^{\wedge k} \rightarrow \mathcal{V}^{\wedge k}$, determined by the equalities

$$A^{\wedge k}(u_1 \wedge \cdots \wedge u_k) = Au_1 \wedge \cdots \wedge Au_k,$$

is the compression $\pi_{\mathcal{V}^{\wedge k}} A^{\otimes k}|_{\mathcal{U}^{\wedge k}}$ of the tensor power $A^{\otimes k} : \mathcal{U}^{\otimes k} \rightarrow \mathcal{V}^{\otimes k}$. Here π_M denotes the orthogonal projection onto a subspace M .

If e_1, \dots, e_n form a basis for \mathcal{V} , then $e_{i_1} \wedge \cdots \wedge e_{i_k}$, $1 \leq i_1 < \cdots < i_k \leq n$, form a basis for $\mathcal{V}^{\wedge k}$ of cardinality $\binom{n}{k}$. Relative to a choice of bases for \mathcal{U} and

\mathcal{V} , the matrix entry for $A^{\wedge k}$ in row-column position $((i_1, \dots, i_k), (j_1, \dots, j_k))$ is the minor of the matrix of A built from rows i_1, \dots, i_k and columns j_1, \dots, j_k .

If \mathcal{U} and \mathcal{V} are normed vector spaces and if $S(z) = \sum_{r \in \mathbb{N}_0^d} S_r z^r$ is a power series with coefficients in $\mathcal{L}(\mathcal{U}, \mathcal{V})$, then, for any tuple $T = (T_1, \dots, T_d)$ of commuting operators on some normed vector space \mathcal{H} , we may consider the operator

$$S(T) = \sum_{r \in \mathbb{N}_0^d} S_r \otimes T^r$$

acting from $\mathcal{U} \otimes \mathcal{H}$ to $\mathcal{V} \otimes \mathcal{H}$, provided the series converges. More generally, starting with power series $S_j(z) = \sum_{r \in \mathbb{N}_0^d} (S_j)_r z^r$, $j = 1, \dots, k$, with coefficients in $\mathcal{L}(\mathcal{U}, \mathcal{V})$, and $T = (T_1, \dots, T_d)$ as above, form the operator

$$(S_1 \otimes \dots \otimes S_k)(T) = \sum_{r_1, \dots, r_k \in \mathbb{N}_0^d} \left(\bigotimes_{j=1}^k (S_j)_{r_j} \right) \otimes T^{r_1 + \dots + r_k},$$

and its compression

$$(S_1 \wedge \dots \wedge S_k)(T) = (\pi_{\mathcal{V}^{\wedge k}} \otimes I_{\mathcal{H}})(S_1 \otimes \dots \otimes S_k)(T)|_{\mathcal{U}^{\wedge k} \otimes \mathcal{H}}.$$

The objective of this section is to establish the following theorem.

Theorem 4.1. *Let \mathcal{U} and \mathcal{V} be Hilbert spaces and let S_1, \dots, S_k belong to $\mathcal{SA}_d(\mathcal{U}, \mathcal{V})$. Then $S_1 \otimes \dots \otimes S_k$ belongs to $\mathcal{SA}_d(\mathcal{U}^{\otimes k}, \mathcal{V}^{\otimes k})$ and $S_1 \wedge \dots \wedge S_k$ belongs to $\mathcal{SA}_d(\mathcal{U}^{\wedge k}, \mathcal{V}^{\wedge k})$.*

Proof. Let $T = (T_1, \dots, T_d)$ be a tuple of commuting strict contractions on some Hilbert space \mathcal{H} . Then the mapping

$$\begin{aligned} \left(\bigotimes_{j=1}^k S_j \right)(T) &= \sum_{r_1, \dots, r_k \in \mathbb{N}_0^d} \left(\bigotimes_{j=1}^k (S_j)_{r_j} \right) \otimes T^{r_1 + \dots + r_k} \\ &= \prod_{j=1}^k \sum_{r_j \in \mathbb{N}_0^d} I_{\mathcal{V}} \otimes \dots \otimes I_{\mathcal{V}} \otimes (S_j)_{r_j} \otimes I_{\mathcal{V}} \otimes \dots \otimes I_{\mathcal{V}} \otimes T^{r_j} \\ &= \prod_{j=1}^k \left(I_{\mathcal{V}} \otimes \dots \otimes I_{\mathcal{V}} \otimes S_j \otimes I_{\mathcal{V}} \otimes \dots \otimes I_{\mathcal{V}} \right)(T) \end{aligned}$$

is contractive as a product of contractive factors. Hence $\bigotimes_{j=1}^k S_j$ belongs to $\mathcal{SA}_d(\mathcal{U}^{\otimes k}, \mathcal{V}^{\otimes k})$. Consequently,

$$\|S_1 \wedge \dots \wedge S_k\|_{\mathcal{A}} \leq \|S_1 \otimes \dots \otimes S_k\|_{\mathcal{A}} \leq 1,$$

which gives the second assertion. \square

Corollary 4.2. *Let S be a $n \times n$ matrix-valued Schur–Agler function, i.e., $S \in \mathcal{SA}_d(\mathbb{C}^n)$. Then, for every $k = 1, \dots, n$, the k -th compound matrix-valued function of S is also Schur–Agler. In particular, $\det S(z)$ is a Schur–Agler function.*

Proof. The matrix of $S^{\wedge k}$ is the k -th compound matrix of S . The case $k = n$ corresponds to $\det S(z)$. \square

Similarly, in the setting of k -th symmetric tensor powers, one may consider the operators $(S_1 \vee \cdots \vee S_k)(T)$. The proof of the following theorem is omitted as it parallels the preceding development.

Theorem 4.3. *Let \mathcal{U} and \mathcal{V} be Hilbert spaces and let S_1, \dots, S_k belong to $\mathcal{SA}_d(\mathcal{U}, \mathcal{V})$. Then $S_1 \vee \cdots \vee S_k$ belongs to $\mathcal{SA}_d(\mathcal{U}^{\vee k}, \mathcal{V}^{\vee k})$.*

Corollary 4.4. *Let S be a $n \times n$ matrix-valued Schur–Agler function, i.e., $S \in \mathcal{SA}_d(\mathbb{C}^n)$. Then, for every $k = 1, \dots, n$, the k -th permanental compound matrix-valued function of S is also Schur–Agler. In particular, the permanent of a Schur–Agler function is also Schur–Agler.*

We note that a permanental analog of (1.1) features in [7].

5. Agler denominators

We are in a position to discuss (eventual) Agler denominators and stability in relation to (1.1). It will first be shown that there exist stable polynomials in three or more variables that are not Agler denominators.

Example 5.1. Let $p(z)$ be a d -variable polynomial, with $\|p\|_\infty = 1$ and multi-degree m , violating the von Neumann inequality (1.4). Let there exist a tuple $T = (T_1, \dots, T_d)$ of commuting contractions such that $T^k = T_1^{k_1} T_2^{k_2} \cdots T_d^{k_d} = 0$, for some $k \in \mathbb{N}_0^d$, and $\|p(T)\| > 1$. Examples of such a scenario can be found in [26, 14, 8]; see also Section 6.

For $0 < r < 1$, the polynomial $q(z) = 1 + rz^{k+m}\bar{p}(1/z)$ is stable and so the rational function

$$f(z) = \frac{z^{k+m} + rp(z)}{1 + rz^{k+m}\bar{p}(1/z)}$$

is inner. However, since $f(T) = rp(T)$, f does not belong to \mathcal{SA}_d whenever $r > 1/\|p(T)\|$. In particular, if the multi-degree of $z^m\bar{p}(1/z)$ is also m , then $f(z) = z^{k+m}\bar{q}(1/z)/q(z)$, so that q is not an Agler denominator.

To give a concrete example, we specialize to the Kaijser–Varopoulos–Holbrook setting. The polynomial

$$p(z_1, z_2, z_3) = \frac{1}{5} \left(z_1^2 + z_2^2 + z_3^2 - 2z_1z_2 - 2z_2z_3 - 2z_3z_1 \right)$$

satisfies $\|p\|_\infty = 1$, and there exist commuting contractions T_1, T_2, T_3 such that $\|p(T_1, T_2, T_3)\| = 6/5$ and $T_1T_2T_3 = 0$. The corresponding rational inner function

$$f(z_1, z_2, z_3) = \frac{z_1^3z_2^3z_3^3 + \frac{r}{5}(z_1^2 + z_2^2 + z_3^2 - 2z_1z_2 - 2z_2z_3 - 2z_3z_1)}{1 + \frac{r}{5}z_1z_2z_3(z_1^2z_2^2 + z_2^2z_3^2 + z_3^2z_1^2 - 2z_1z_2z_3^2 - 2z_1z_3^2z_2 - 2z_1^2z_2z_3)},$$

is not Schur–Agler for $5/6 < r < 1$. For these values of r , the stable polynomial

$$q(z_1, z_2, z_3) = 1 + \frac{r}{5} z_1 z_2 z_3 \left(z_1^2 z_2^2 + z_2^2 z_3^2 + z_3^2 z_1^2 - 2z_1 z_2 z_3^2 - 2z_1 z_2^2 z_3 - 2z_1^2 z_2 z_3 \right)$$

is not an Agler denominator.

We now have the following result.

Theorem 5.2. *Let a polynomial p admit a representation (1.1) for some $n \in \mathbb{N}_0^d$ and contractive K . Then*

$$\frac{z^n \bar{p}(1/z)}{p(z)} = \det(-K^* + \sqrt{I - K^* K} Z_n (I - K Z_n)^{-1} \sqrt{I - K K^*}). \quad (5.1)$$

In particular, p is an eventual Agler denominator of order n . If $\deg p = n$, then p is an Agler denominator.

A lemma is needed; see, e.g., [23, Theorem 3.1.2].

Lemma 5.3. *Let A, B, C , and D be square matrices of the same size, and suppose that $AC = CA$. Then*

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(AD - CB).$$

Proof of Theorem 5.2. By Lemma 2.2, the right hand side of (5.1) equals

$$\frac{\det \begin{bmatrix} -K^* & -\sqrt{I - K^* K} Z_n \\ \sqrt{I - K K^*} & I - K Z_n \end{bmatrix}}{\det(I - K Z_n)}. \quad (5.2)$$

Let $K = U \Sigma V^*$ be a singular value decomposition of K . Then the numerator of (5.2) equals

$$\det \begin{bmatrix} V U^* & 0 \\ 0 & I \end{bmatrix} \det \begin{bmatrix} -U \Sigma U^* & -U \sqrt{I - \Sigma^2} V^* Z_n \\ U \sqrt{I - \Sigma^2} U^* & I - K Z_n \end{bmatrix}. \quad (5.3)$$

Applying Lemma 5.3, noting that $-U \Sigma U^*$ and $U \sqrt{I - \Sigma^2} U^*$ commute, we get that (5.3) equals

$$\begin{aligned} \det(V U^*) \det(-U \Sigma U^* (I - U \Sigma V^* Z_n) + U \sqrt{I - \Sigma^2} U^* U \sqrt{I - \Sigma^2} V^* Z_n) \\ = \det(Z_n - K^*). \end{aligned} \quad (5.4)$$

To prove (5.1) it remains to observe that

$$\begin{aligned} z^n \bar{p}(1/z) &= z^n \det(I - \bar{K} Z_n^{-1}) = \det(I - \bar{K} Z_n^{-1}) \det Z_n \\ &= \det(Z_n - \bar{K}) = \det(Z_n - K^*), \end{aligned} \quad (5.5)$$

where in the last step we used that $Z_n^\top = Z_n$. As the Julia operator

$$\begin{bmatrix} -K^* & \sqrt{I - K^* K} \\ \sqrt{I - K K^*} & K \end{bmatrix}$$

is unitary, the multivariable rational inner matrix function

$$-K^* + \sqrt{I - K^* K} Z_n (I - K Z_n)^{-1} \sqrt{I - K K^*}$$

is in the Schur–Agler class. By Corollary 4.2, so is its determinant, and thus $z^n \bar{p}(1/z)/p(z)$ is in the Schur–Agler class. \square

Corollary 5.4. *For every $p \in \mathbb{C}[z_1, \dots, z_d]$ with $p(0) = 1$, there exists $r > 0$ such that the polynomial $p_r(z) := p(rz)$ is an eventual Agler denominator. In fact, if p is given by (1.1), then one can choose any $0 < r \leq 1/\|K\|$.*

Proof. Since, by Theorem 2.1, every polynomial p with $p(0) = 1$ admits a representation (1.1), the assertion follows from the identity

$$p_r(z) = \det(I_{|n|} - rKZ_n),$$

and Theorem 5.2. \square

Remark 5.5. The first statement of Corollary 5.4 can also be deduced from Corollary 3.3: since $\|1 - p(0)\|_{\mathcal{A}} = 0$, the inequality $\|1 - p_r\|_{\mathcal{A}} \leq 1$ holds, by continuity, for a sufficiently small $r > 0$. For multi-affine symmetric polynomials a stronger statement is true [18, Theorem 1.5]: p_r is an Agler denominator for sufficiently small $r > 0$.

Following [3], we call a semi-stable polynomial p *scattering Schur* if p and $z^{\deg p} \bar{p}(1/z)$ have no factor in common. In [19, Theorem 1] it was proven that every two-variable scattering Schur polynomial p of degree $n = (n_1, n_2)$ is of the form (1.1) with K an $(n_1 + n_2) \times (n_1 + n_2)$ contraction. Thus every two-variable scattering Schur polynomial p is an Agler denominator.

The following result provides a partial converse to Theorem 5.2.

Theorem 5.6. *Let p be a d -variable scattering Schur polynomial with $p(0) = 1$. If, for some $m \in \mathbb{N}_0^d$, the rational inner function $z^m \bar{p}(1/z)/p(z)$ has a transfer-function realization (1.5) of order m , then p admits a representation (1.1) with $n = m$ and K a contraction.*

Proof. Taking the determinant of both sides of the equality

$$\frac{z^m \bar{p}(1/z)}{p(z)} = A + BZ_m(I - DZ_m)^{-1}C$$

and using Lemma 2.2, we obtain

$$\frac{z^m \bar{p}(1/z)}{p(z)} = \frac{\det \begin{bmatrix} A & -BZ_m \\ C & I - DZ_m \end{bmatrix}}{\det(I - DZ_m)} =: \frac{r(z)}{s(z)}.$$

Note that both $r(z)$ and $s(z)$ are of degree at most m . We now obtain that

$$(z^m \bar{p}(1/z))s(z) = r(z)p(z).$$

As p is scattering Schur we must have that $p(z)$ divides $s(z)$, say $s(z) = q(z)p(z)$. Dividing out $p(z)$ in the above equation, we obtain that

$$(z^m \bar{p}(1/z))q(z) = r(z).$$

As the left hand side has degree $m + \deg q$ and the right hand side degree at most m , we obtain that q must be a constant. But then, using $p(0) = 1$ and $s(0) = \det(I - DZ_m)|_{z=0} = 1$, we obtain that $q = 1$, and thus $p(z) = \det(I - KZ_m)$ with $K = D$. \square

Corollary 5.7. *The polynomials $p(z_1, \dots, z_d) = 1 - \sum_{i=1}^d a_i z_i$ with $\sum_{i=1}^d |a_i| \leq 1$, are Agler denominators.*

Proof. Let K be a $d \times d$ rank 1 contraction with diagonal entries a_1, \dots, a_d . One such choice is given by

$$K = \left[\sqrt{|a_j a_k|} e^{i \arg a_k} \right]_{j,k=1}^d.$$

Then $\det(I_d - KZ_{(1,\dots,1)}) = p(z)$ and the result follows directly from Theorem 5.2. \square

Remark 5.8. The matrix K in the proof of Corollary 5.7 is clearly of minimal size. It is also of minimal norm, $\|K\| = |a_1| + \dots + |a_d|$, for otherwise $p(z) = \det(I_d - KZ_{(1,\dots,1)})$ would be stable.

It was shown in [18, Theorem 3.3] that a multi-affine symmetric polynomial is an Agler denominator if and only if a certain matrix \mathcal{B} constructed from the Christoffel–Darboux equation is positive semidefinite. Subsequently, for $p(z) = 1 - \frac{1}{d} \sum_{i=1}^d z_i$, the positivity of the matrix \mathcal{B} was computationally checked up to $d = 11$. Using our Corollary 5.7, we deduce this fact for all d .

Corollary 5.9. *Let $p(z) = t - \frac{1}{d} \sum_{i=1}^d z_i$, where $|t| \geq 1$, and let*

$$\mathcal{B} := (B_{|\alpha|,|\beta|}^{|\alpha \cap \beta|})_{\alpha, \beta \subseteq \{1, \dots, d-1\}},$$

be the $2^{(d-1) \times (d-1)}$ matrix indexed by subsets α, β of $\{1, \dots, d-1\}$, defined via:

$$\binom{d}{j}^{-1} \binom{d}{k}^{-1} (p_j \bar{p}_k - \bar{p}_{d-j} p_{d-k}) = (d-j-k+i) B_{j,k}^i - i B_{j-1,k-1}^{i-1}, \quad (5.6)$$

where $0 \leq i \leq j, k \leq d-1, p_0 = t, p_1 = \frac{1}{d}, p_j = 0, j \geq 2$, and $B_{j,k}^i = 0$ for i, j, k not satisfying $0 \leq i \leq j, k \leq d-1$. Then \mathcal{B} is positive semidefinite.

To illustrate, we choose t real and $d = 3$:

$$\mathcal{B} = \frac{1}{18} \begin{bmatrix} 6t^2 & -3t & -3t & 0 \\ -3t & 3t^2 + 1 & 2 & -3t \\ -3t & 2 & 3t^2 + 1 & -3t \\ 0 & -3t & -3t & 6t^2 \end{bmatrix} = A^* A,$$

where

$$A = \frac{1}{\sqrt{18}} \begin{bmatrix} \sqrt{6}t & -\frac{1}{2}\sqrt{6} & -\frac{1}{2}\sqrt{6} & 0 \\ 0 & -\frac{1}{2}\sqrt{6} & -\frac{1}{2}\sqrt{6} & \sqrt{6}t \\ 0 & \sqrt{3(t^2-1)} & \sqrt{3(t^2-1)} & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}.$$

Remark 5.10. In the context of (1.1), the question of whether a given polynomial is an (eventual) Agler denominator is reduced to analyzing the matrix of its determinantal representation. This provides a possible alternative to the

transfer-function realization method. For example, $p(z) = 1 - \frac{1}{3}(z_1 + z_2 + z_3)$ admits a representation (1.1) with

$$K = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

which is minimal both in size and in norm; see Remark 5.8. At the same time, the minimal order of a transfer-function realization (1.5) of

$$z_1 z_2 z_3 \frac{\bar{p}(1/z)}{p(z)} = \frac{3z_1 z_2 z_3 - z_2 z_3 - z_1 z_3 - z_1 z_2}{3 - z_1 - z_2 - z_3}$$

is $m = (2, 2, 2)$ [17, 5].

6. Variations on the Kaijser–Varopoulos–Holbrook example

For s real, consider the multivariable polynomial

$$p(z_1, \dots, z_d) = (1 + s) \sum_{m=1}^d z_m^2 - \left(\sum_{m=1}^d z_m \right)^2.$$

The case of $d = 3$ and $s = 1$ corresponds to the polynomial from [26, 14].

Proposition 6.1. *Let $d > 1$, $s > d/2 - 1$, and p be defined as above. Then $\|p\|_\infty = (1 + s)d$, if d is even, and $\|p\|_\infty < (1 + s)d$, if d is odd, while $\|p\|_A = (1 + s)d$ for all d . In particular, $p/\|p\|_\infty$ is not in \mathcal{SA}_d for odd $d > 1$.*

Proof. Write $p(z_1, \dots, z_d) = z^\top A z$, where

$$A = \begin{bmatrix} s & -1 & \dots & -1 \\ -1 & s & \dots & -1 \\ & & \ddots & \\ -1 & -1 & \dots & s \end{bmatrix} \quad \text{and} \quad z = \begin{bmatrix} z_1 \\ \vdots \\ z_d \end{bmatrix}.$$

Observe that A is symmetric with eigenvalues $1 + s$ and $s - d + 1$, so

$$\|A\| = \max \{1 + s, |s - d + 1|\} = 1 + s.$$

Hence $\|p\|_\infty \leq (1 + s)d$. If d is even, one immediately has equality since $p(1, -1, \dots, 1, -1) = (1 + s)d$. If d is odd, the inequality is strict. Indeed, otherwise $|p|$ would be maximized for some unimodular z_1, \dots, z_d with zero sum, as z would then lie in the eigenspace of A corresponding to $1 + s$. But then the equality

$$|p(z_1, \dots, z_d)| = (1 + s) \left| \sum_{i=1}^d z_i^2 \right| = (1 + s)d$$

would force $z_i = \pm e^{i\alpha}$, $i = 1, \dots, d$, in conflict with the zero sum condition.

Next, for a tuple of commuting contractions $T = (T_1, \dots, T_d)$, we have

$$p(T_1, \dots, T_d) = \begin{bmatrix} T_1 & \dots & T_d \end{bmatrix} (A \otimes I) \begin{bmatrix} T_1 \\ \vdots \\ T_d \end{bmatrix},$$

so that

$$\|p(T_1, \dots, T_d)\| \leq \|A\|d = (1+s)d.$$

Choose v_1, \dots, v_d to be any unit vectors in \mathbb{R}^2 with zero sum. Then the matrices

$$T_i = \begin{bmatrix} 0 & v_i^\top & 0 \\ 0 & 0 & v_i \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{4 \times 4}, \quad i = 1, \dots, d,$$

are such that $\|T_i\| = 1$, $T_i T_j = T_j T_i = \langle v_i, v_j \rangle e_1 e_4^\top$, $\sum_{i=1}^d T_i = 0$, and

$$p(T_1, \dots, T_d) e_4 = (1+s) d e_1,$$

where e_j is the j th standard unit vector in \mathbb{R}^4 . Hence $\|p\|_{\mathcal{A}} = (1+s)d$. \square

Remark 6.2. In the case of $d = 3$, maximizing $\|p\|_{\mathcal{A}}/\|p\|_{\infty}$, the von Neumann constant of p , over s , we find that the maximum possible ratio is $\frac{1}{3}\sqrt{\frac{35+13\sqrt{13}}{6}} \approx 1.23$ (occurring for $s = \frac{\sqrt{13}+1}{6}$). The previously known lower bound for the von Neumann constant was $\frac{6}{5}$ [14].

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